

# Duality in Auslander's $k$ -Gorenstein rings <sup>\*†</sup>

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## Abstract

In this paper we mainly study the properties of dual modules over  $k$ -Gorenstein rings. For a  $k$ -Gorenstein ring  $\Lambda$  we show that the right self-injective dimension is less than or equal to  $k$  if and only if each finitely generated left  $\Lambda$ -module  $M$  with the property that  $\text{Ext}_\Lambda^i(M, \Lambda) = 0$  for any  $1 \leq i \leq k$  is reflexive. We claim that the big and small left finitistic dimensions and left self-injective dimension of an Auslander ring are identical, from which we get some equivalent conditions of the famous Nakayama conjecture and then give a partial answer to it. Moreover, we show that if  $\Lambda$  is an Auslander-Gorenstein ring then  $\text{Ext}_\Lambda^{\text{grade } M}(M, \Lambda)$  is pure for any  $M \in \text{mod } \Lambda$ . This result answers an open question of Björk affirmatively. For a 2-Gorenstein ring  $\Lambda$  we show that a non-zero proper left ideal  $I$  of  $\Lambda$  is reflexive if and only if  $\Lambda/I$  has no non-zero pseudo-null submodule.

## 1. Introduction

A left and right noetherian ring  $\Lambda$  is called  $k$ -Gorenstein if the left flat dimension of the  $i$ th term in a minimal injective resolution of  $\Lambda$  as a left  $\Lambda$ -module is less than or equal to  $i - 1$  for any  $1 \leq i \leq k$ , and  $\Lambda$  is called an Auslander ring if it is  $k$ -Gorenstein for all  $k \geq 1$ . We shall say that any left and right noetherian ring is 0-Gorenstein. Auslander gave some equivalent conditions of  $k$ -Gorenstein rings in terms of the strong grade of modules (see section 2 for the definition) and right flat dimension as follows.

**Auslander's Theorem** ([9] Theorem 3.7) *The following statements are equivalent for a left and right noetherian ring  $\Lambda$ .*

- (1)  $\Lambda$  is a  $k$ -Gorenstein ring.
- (2) *The right flat dimension of the  $i$ th term in a minimal injective resolution of  $\Lambda$  as a right  $\Lambda$ -module is less than or equal to  $i - 1$  for any  $1 \leq i \leq k$ .*

(3)  $\text{s.grade Ext}_\Lambda^i(M, \Lambda) \geq i$  for any finitely generated left  $\Lambda$ -module  $M$  and  $1 \leq i \leq k$ .

(4)  $\text{s.grade Ext}_\Lambda^i(N, \Lambda) \geq i$  for any finitely generated right  $\Lambda$ -module  $N$  and  $1 \leq i \leq k$ .

Auslander's Theorem is a very useful characterization of  $k$ -Gorenstein rings, especially it shows the notion of  $k$ -Gorenstein rings is left-right symmetric.  $k$ -Gorenstein rings has been studied by

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several authors (see [3], [7-9], [17] and [19], etc). In this paper we mainly investigate the properties of dual modules over  $k$ -Gorenstein rings. We find that many properties of modules have good behavior in such rings.

From [15] Theorem 2.2 we know that if the right self-injective dimension of a left and right noetherian ring  $\Lambda$  is less than or equal to  $k$  then each finitely generated left  $\Lambda$ -module  $M$  with the property that  $\text{Ext}_{\Lambda}^i(M, \Lambda) = 0$  for any  $1 \leq i \leq k$  is reflexive. In section 2 we show that the converse of this result also holds for  $k$ -Gorenstein rings. It is showed in [21] that the big finitistic dimension and the small finitistic dimension of  $\Lambda$  are usually different even when  $\Lambda$  is an artin algebra. The another main result of section 2 is to show that for a  $(k+1)$ -Gorenstein ring  $\Lambda$  the small left finitistic dimension of  $\Lambda$  is less than or equal to  $k$  if and only its left self-injective dimension is less than or equal to  $k$ , which yields that the big and small left finitistic dimensions and the left self-injective dimension of  $\Lambda$  are identical if  $\Lambda$  is an Auslander ring. We then get that the famous Nakayama conjecture holds if and only if a finitely generated left (simple)  $\Lambda$ -module with infinite grade is zero for an artin algebra  $\Lambda$  with infinite dominant dimension. At the end of this section, we give a partial answer to this conjecture. Let  $\Lambda$  be an Auslander-Gorenstein ring. Is it true that  $\text{Ext}_{\Lambda}^{\text{grade } M}(M, \Lambda)$  is pure for each finitely generated left  $\Lambda$ -module  $M$ ? This is an open question raised by Björk in 1989. In section 3 we give a complete affirmative answer to this question. We also give in this section some useful properties of  $k$ -Gorenstein rings and Auslander rings or concerning such rings. We give in the final section a criterion for judging when a finitely generated torsionless module is reflexive. As an immediate corollary of the criterion we have that for a 2-Gorenstein ring  $\Lambda$  a non-zero proper left ideal  $I$  of  $\Lambda$  is reflexive if and only if  $\Lambda/I$  has no non-zero pseudo-null submodule, which generalizes a result by Coates, Schneider and Sujatha. Ramras in 1972 raised an open question as follows. For a left and right noetherian ring  $\Lambda$ , when is each reflexive in  $\text{mod } \Lambda$  projective? Also in this section we study a generalized version of this question. We claim that the answer to this question is positive if the global dimension of  $\Lambda$  is less than or equal to two, which give a partial answer to the question of Ramras.

In the following, we will feel free to use Auslander's Theorem without mentioning it. Unless stated otherwise,  $\Lambda$  is a left and right noetherian ring. We use  $\text{mod } \Lambda$  to denote the category of finitely generated left  $\Lambda$ -modules. For a left (resp. right)  $\Lambda$ -module  $M$ , we use  $\text{l.id}_{\Lambda} M$  (resp.  $\text{r.id}_{\Lambda} M$ ) to denote the left (resp. right) injective dimension of  $M$ , and use  $\text{l.pd}_{\Lambda} M$  (resp.  $\text{r.pd}_{\Lambda} M$ ) to denote the left (resp. right) projective dimension of  $M$ . In addition, we assume that

$$0 \rightarrow \Lambda \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_i \rightarrow \cdots$$

is a minimal injective resolution of  $\Lambda$  as a right  $\Lambda$ -module.

## 2. Reflexive modules and homological dimensions

Let  $M$  be in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ). For a non-negative integer  $t$ ,  $M$  is called a  $W^t$ -module if  $\text{Ext}_\Lambda^i(M, \Lambda) = 0$  for any  $1 \leq i \leq t$ ; and  $M$  is called a  $W^\infty$ -module if it is a  $W^t$ -module for all  $t$ . If the notation in [2] is adopted, we remark that a module is a  $W^\infty$ -module if and only if it is in  ${}^\perp \Lambda$ . We denote  $\text{grade} M = t$  if  $\text{Ext}_\Lambda^i(M, \Lambda) = 0$  for any  $0 \leq i < t$  and  $\text{Ext}_\Lambda^t(M, \Lambda) \neq 0$ , and denote  $\text{s.grade} M = t$  if  $\text{grade} A = t$  for each submodule of  $M$ . Moreover, if  $\text{Ext}_\Lambda^i(M, \Lambda) = 0$  for any  $i \geq 0$  we then write  $\text{grade} M = \infty$ . Let  $\sigma_M : M \rightarrow M^{**}$  via  $\sigma_M(x)(f) = f(x)$  for any  $x \in M$  and  $f \in M^*$  be the canonical evaluation homomorphism, where  $( )^* = \text{Hom}(-, \Lambda)$ .  $M$  is called a torsionless module if  $\sigma_M$  is a monomorphism; and  $M$  is called a reflexive module if  $\sigma_M$  is an isomorphism.

For a positive integer  $k$ , we know from [15] Theorem 2.2 that if  $\text{r.id}_\Lambda \Lambda \leq k$  then each  $W^k$ -module in  $\text{mod } \Lambda$  is reflexive. However, we don't know whether the converse holds. Jans showed in [18] Theorem 5.1 that the converse is true in the case  $k = 1$ . Here we give the following result.

**Theorem 2.1** *Let  $k$  be a non-negative integer and  $\text{grade Ext}_\Lambda^i(N, \Lambda) \geq i$  for any  $N \in \text{mod } \Lambda^{op}$  and  $1 \leq i \leq k$  (especially,  $\Lambda$  a  $k$ -Gorenstein ring). Then the following statements are equivalent.*

- (1)  $\text{r.id}_\Lambda \Lambda \leq k$ .
- (2) Each  $W^k$ -module in  $\text{mod } \Lambda$  is reflexive.
- (3) Each  $W^k$ -module in  $\text{mod } \Lambda$  is torsionless.

Before proving this theorem, we first recall some notions from [2] and [18]. Let  $M$  be in  $\text{mod } \Lambda$  and  $\mathcal{D}$  a full subcategory of  $\text{mod } \Lambda$ . A homomorphism  $M \xrightarrow{f} D$  with  $D \in \mathcal{D}$  is called a left  $\mathcal{D}$ -approximation of  $M$  if  $\text{Hom}_\Lambda(f, D')$  is epimorphic for any  $D' \in \mathcal{D}$ . A monomorphism  $X^{**} \xrightarrow{\rho^*} Y^*$  in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) is called a double dual embedding if it is the dual of an epimorphism  $Y \xrightarrow{\rho} X^*$  in  $\text{mod } \Lambda^{op}$  (resp.  $\text{mod } \Lambda$ ). For a positive integer  $k$ , a torsionless module  $T_k$  in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) is said to be of  $D$ -class  $k$  if it can be fitted into an exact sequence of the form:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_{k-1}^{**} & \longrightarrow & P_{k-1} & \longrightarrow & T_k \longrightarrow 0 \\
 & & \uparrow \sigma_{T_{k-1}} & & & & \\
 \cdots & \longrightarrow & P_{k-2} & \longrightarrow & T_{k-1} & \longrightarrow & 0 \\
 & & & & & & \\
 & & \cdots & & & & \\
 & & & & & & \\
 0 & \longrightarrow & T_2^{**} & \longrightarrow & \cdots & & \\
 & & \uparrow \sigma_{T_2} & & & & \\
 0 & \longrightarrow & T_1^{**} & \longrightarrow & P_1 & \longrightarrow & T_2 \longrightarrow 0
 \end{array}$$

where each  $P_i$  is projective in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) and the horizontal monomorphisms are double dual embeddings. Any torsionless module is said to be of  $D$ -class 1. From [18] p.335 we also know that a torsionless module in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) is of  $D$ -class  $k$  if it is a  $W^{k-1}$ -module. The following are two main results of [18], which are crucial for proving Theorem 2.1.

**Lemma 2.2** ([18] Theorem 4.3)  *$\text{r.id}_\Lambda \Lambda \leq k$  if and only if each module of  $D$ -class  $k$  in  $\text{mod } \Lambda$  is reflexive.*

The proof of [18] Theorem 3.1 in fact proves the following more general result.

**Lemma 2.3** *If  $\text{gradeExt}_\Lambda^i(N, \Lambda) \geq i$  for any  $N \in \text{mod } \Lambda^{op}$  and  $1 \leq i \leq k$ , then each module of  $D$ -class  $k$  in  $\text{mod } \Lambda$  is a  $W^{k-1}$ -module.*

*Proof of Theorem 2.1.* The case  $k = 0$  follows from [15] Lemma 2.1. Now suppose  $k \geq 1$ . It suffices to prove (3) implies (1). Assume that (3) holds and  $M$  is a module of  $D$ -class  $k$  in  $\text{mod } \Lambda$ . Then  $M$  is torsionless. By Lemma 2.3, we have that  $M$  is a  $W^{k-1}$ -module.

Let  $P \xrightarrow{f} M^* \rightarrow 0$  be exact in  $\text{mod } \Lambda^{op}$  with  $P$  projective. Put  $g = f^* \sigma_M$  and  $X = \text{Cokerg}$ . It is not difficult to verify that  $g : M \rightarrow P^*$  is a left  $\mathcal{P}^0(\Lambda)$ -approximation of  $M$  (where  $\mathcal{P}^0(\Lambda)$  denotes the full subcategory of  $\text{mod } \Lambda$  consisting of projective modules). Then  $X$  is a  $W^1$ -module and so  $X$  is a  $W^k$ -module. By (3) we have that  $X$  is torsionless and  $\sigma_X$  is a monomorphism. On the other hand, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{g} & P^* & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow \sigma_M & & \downarrow \cong & & \downarrow \sigma_X & & \\ 0 & \longrightarrow & M^{**} & \xrightarrow{g^{**}} & P^{***} & \longrightarrow & X^{**} & & \end{array}$$

By the Snake lemma,  $\sigma_M$  is an isomorphism and  $M$  is reflexive. It then follows from Lemma 2.2 that  $\text{r.id}_\Lambda \Lambda \leq k$ . ■

Notice that the notion of  $k$ -Gorenstein rings is left-right symmetric. So we have the following dual statement of Theorem 2.1.

**Theorem 2.4** *Let  $k$  be a non-negative integer and  $\text{gradeExt}_\Lambda^i(M, \Lambda) \geq i$  for any  $M \in \text{mod } \Lambda$  and  $1 \leq i \leq k$  (especially,  $\Lambda$  a  $k$ -Gorenstein ring). Then the following statements are equivalent.*

- (1)  $\text{l.id}_\Lambda \Lambda \leq k$ .
- (2) Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is reflexive.
- (3) Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is torsionless.

The statement (1) of the following lemma is just [14] Corollary 3, and the statement (2) is the dual of (1).

**Lemma 2.5** *Let  $\Lambda$  be an artin ring.*

(1) If  $\text{r.id}_\Lambda \Lambda = k$  and the first  $k$  terms in a minimal injective resolution of  $\Lambda$  as a right  $\Lambda$ -module have finite right flat dimension, then  $\text{l.id}_\Lambda \Lambda = k$ .

(2) If  $\text{l.id}_\Lambda \Lambda = k$  and the first  $k$  terms in a minimal injective resolution of  $\Lambda$  as a left  $\Lambda$ -module have finite left flat dimension, then  $\text{r.id}_\Lambda \Lambda = k$ .

Since for a  $k$ -Gorenstein ring  $\Lambda$  the  $i$ th term in a minimal injective resolution of  $\Lambda$  as a left  $\Lambda$ -module has left flat dimension less than or equal to  $i - 1$  for any  $1 \leq i \leq k$  and also since the left-right symmetry of the notion of  $k$ -Gorenstein rings, as an immediate result of Lemma 2.5 we have

**Corollary 2.6**  $\text{r.id}_\Lambda \Lambda \leq k$  if and only if  $\text{l.id}_\Lambda \Lambda \leq k$  for an artin  $k$ -Gorenstein ring  $\Lambda$ .

The following result generalizes [3] Corollary 5.5(b).

**Corollary 2.7**  $\text{r.id}_\Lambda \Lambda = \text{l.id}_\Lambda \Lambda$  for an Auslander artin ring  $\Lambda$ .

**Corollary 2.8** Let  $\Lambda$  be an artin  $k$ -Gorenstein ring. Then the following statements are equivalent.

- (1)  $\text{r.id}_\Lambda \Lambda \leq k$ .
- (2) Each  $W^k$ -module in  $\text{mod } \Lambda$  is reflexive.
- (3) Each  $W^k$ -module in  $\text{mod } \Lambda$  is torsionless.
- (1)<sup>op</sup>  $\text{l.id}_\Lambda \Lambda \leq k$ .
- (2)<sup>op</sup> Each  $W^k$ -module in  $\text{mod } \Lambda^{\text{op}}$  is reflexive.
- (3)<sup>op</sup> Each  $W^k$ -module in  $\text{mod } \Lambda^{\text{op}}$  is torsionless.

*Proof.* By Theorems 2.1 and 2.4 and Corollary 2.6. ■

The big left finitistic dimension of  $\Lambda$ , written  $\text{lFin.dim } \Lambda$ , is defined to be  $\sup\{\text{l.pd}_\Lambda M \mid M \text{ is a left } \Lambda\text{-module with } \text{l.pd}_\Lambda M < \infty\}$ ; and the small left finitistic dimension of  $\Lambda$ , written  $\text{lfin.dim } \Lambda$ , is defined to be  $\sup\{\text{l.pd}_\Lambda M \mid M \text{ is in } \text{mod } \Lambda \text{ with } \text{l.pd}_\Lambda M < \infty\}$ . Dually, the big and small right finitistic dimensions of  $\Lambda$  are defined, which are denoted by  $\text{rFin.dim } \Lambda$  and  $\text{rfin.dim } \Lambda$  respectively. According to [21] we know that the big and small left (resp. right) finitistic dimensions aren't identical in general even for artin algebras. However, we show here that either of the big and small left (resp. right) finitistic dimensions is equal to the left (resp. right) self-injective dimension for an Auslander ring. As an application to this result, we then give some equivalent conditions of the famous Nakayama conjecture.

We first give the following easy observation.

**Lemma 2.9** If  $M$  is a  $W^k$ -module in  $\text{mod } \Lambda$  with  $\text{l.pd}_\Lambda M \leq k - 1$ , then  $M$  is projective.

*Proof.* Consider the projective resolution of  $M$ , we get easily our assertion. ■

**Lemma 2.10** ([18] Theorem 4.2) For a positive integer  $k$ ,  $\text{lfin.dim } \Lambda \leq k$  if and only if a module

$N$  of  $D$ -class  $k$  in  $\text{mod } \Lambda^{op}$  is projective provided  $N^*$  is projective.

**Lemma 2.11** *Let  $\Lambda$  be a 1-Gorenstein ring. Then  $\text{lfin.dim} \Lambda = 0$  if and only if  $\Lambda$  is (left) self-injective.*

*Proof.* We only need to prove the necessity. Suppose  $\text{lfin.dim} \Lambda = 0$ . We claim that  $\text{Ext}_\Lambda^1(M, \Lambda) = 0$  for any  $M \in \text{mod } \Lambda$ . Otherwise, by [5] Corollary 5.6 and Theorem 5.4 we have  $[\text{Ext}_\Lambda^1(M, \Lambda)]^* \neq 0$ . But  $\Lambda$  is 1-Gorenstein, it then turns out that  $\text{s.grade} \text{Ext}_\Lambda^1(M, \Lambda) \geq 1$ , which induces a contradiction. We are done. ■

Let  $M$  be in  $\text{mod } \Lambda$  and  $k$  a positive integer. We call  $M$  a  $k$ -syzygy module if there is an exact sequence  $0 \rightarrow M \rightarrow Q_0 \rightarrow Q_1 \rightarrow \cdots \rightarrow Q_{k-1}$  in  $\text{mod } \Lambda$  with all  $Q_i$  projective. On the other hand, assume that  $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$  is a projective resolution of  $M$  in  $\text{mod } \Lambda$ . Then we get an exact sequence  $0 \rightarrow M^* \rightarrow P_0^* \xrightarrow{f^*} P_1^* \rightarrow X \rightarrow 0$ , where  $X = \text{Coker } f^*$ .  $M$  is called a  $k$ -torsionfree module if  $X$  is a  $W^k$ -module (see [4]);  $M$  is called a  $\infty$ -torsionfree module if  $X$  is a  $W^\infty$ -module. We remark that it follows from [15] Lemma 2.1 that a module in  $\text{mod } \Lambda$  is torsionless (resp. reflexive) if and only if it is 1-torsionfree (resp. 2-torsionfree).

**Theorem 2.12** *Let  $k$  be a non-negative integer and  $\Lambda$  a  $(k+1)$ -Gorenstein ring with  $\text{lfin.dim} \Lambda = k$ . Then  $\text{l.id}_\Lambda \Lambda \leq k$ .*

*Proof.* The case  $k = 0$  follows from Lemma 2.11. Now suppose  $k \geq 1$  and  $M$  is any module in  $\text{mod } \Lambda$ . Since  $\Lambda$  is  $(k+1)$ -Gorenstein,  $\text{s.grade} \text{Ext}_\Lambda^{k+1}(M, \Lambda) \geq k+1$ . Let

$$Q_{k+1} \rightarrow Q_k \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \text{Ext}_\Lambda^{k+1}(M, \Lambda) \rightarrow 0$$

be a projective resolution of  $\text{Ext}_\Lambda^{k+1}(M, \Lambda)$  in  $\text{mod } \Lambda^{op}$ . Put  $T_k = \text{Coker}(Q_{k+1} \rightarrow Q_k)$  and  $X_k = \text{Coker}(Q_k^* \rightarrow Q_{k+1}^*)$ . Notice that a  $k$ -syzygy module is  $k$ -torsionfree by [4] Proposition 1.6. So  $T_k$  is  $k$ -torsionfree and  $X_k$  is a  $W^k$ -module. On the other hand, we have the following exact sequences:

$$0 \rightarrow T_k^* \rightarrow Q_k^* \rightarrow Q_{k+1}^* \rightarrow X_k \rightarrow 0$$

and

$$0 \rightarrow Q_0^* \rightarrow Q_1^* \rightarrow \cdots \rightarrow Q_k^* \rightarrow Q_{k+1}^* \rightarrow X_k \rightarrow 0.$$

From the last exact sequence we have  $\text{l.pd}_\Lambda X_k \leq k+1$ . But  $\text{lfin.dim} \Lambda = k$ , so  $\text{l.pd}_\Lambda X_k \leq k$  and hence  $X_k$  is a  $W^\infty$ -module, which implies that  $\text{l.pd}_\Lambda T_k^* \leq k-2$  and  $T_k^*$  is also a  $W^\infty$ -module. By Lemma 2.9,  $T_k^*$  is projective. In addition, also because  $T_k^*$  is a  $W^\infty$ -module, we have that  $T_k^*$  is of  $D$ -class  $n$  in  $\text{mod } \Lambda$  for all  $n$ , especially, is of  $D$ -class  $k$ . By Lemma 2.10,  $T_k$  is projective and  $\text{r.pd}_\Lambda \text{Ext}_\Lambda^{k+1}(M, \Lambda) \leq k$ .

We now claim that  $\text{Ext}_\Lambda^{k+1}(M, \Lambda) = 0$ . Assume that  $\text{r.pd}_\Lambda \text{Ext}_\Lambda^{k+1}(M, \Lambda) = t(\leq k)$ . It is not difficult to verify that  $\text{Ext}_\Lambda^t(\text{Ext}_\Lambda^{k+1}(M, \Lambda), \Lambda) \neq 0$ , which induces a contradiction for  $\text{s.gradeExt}_\Lambda^{k+1}(M, \Lambda) \geq k + 1$ . So we conclude that  $\text{l.id}_\Lambda \leq k$ . ■

It is well known that  $\text{lfin.dim} \Lambda \leq \text{lFin.dim} \Lambda \leq \text{l.id}_\Lambda \Lambda$  for any left noetherian ring  $\Lambda$  (c.f. [6] Proposition 4.3). Summarizing the results obtained here we then get the following

**Theorem 2.13** *Let  $k$  be a non-negative integer and  $\Lambda$  a  $(k+1)$ -Gorenstein ring. Then the following statements are equivalent.*

- (1)  $\text{l.id}_\Lambda \leq k$ .
- (2)  $\text{lFin.dim} \Lambda \leq k$ .
- (3)  $\text{lfin.dim} \Lambda \leq k$ .
- (4) Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is reflexive.
- (5) Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is torsionless.
- (6) Each module of  $D$ -class  $k$  in  $\text{mod } \Lambda^{op}$  is reflexive.
- (7) A module  $N$  of  $D$ -class  $k$  in  $\text{mod } \Lambda^{op}$  is projective provided  $N^*$  is projective.

For each positive integer  $n$ , Zimmermann Huisgen in [21] showed that there exists a finite dimensional monomial relation algebra  $\Lambda$  with  $J^4 = 0$  (where  $J$  is the Jacobson Radical of  $\Lambda$ ) such that  $\text{lfin.dim} \Lambda = n + 1$  and  $\text{lFin.dim} \Lambda = n + 2$ . Smalø in [20] constructed examples with arbitrarily big differences between  $\text{lfin.dim} \Lambda$  and  $\text{lFin.dim} \Lambda$ . But very little appears to be known on when  $\text{lfin.dim} \Lambda = \text{lFin.dim} \Lambda$ . As an immediate consequence of Theorem 2.13 we have

**Corollary 2.14**  $\text{lfin.dim} \Lambda = \text{lFin.dim} \Lambda = \text{l.id}_\Lambda \Lambda$  for an Auslander ring  $\Lambda$ .

**Corollary 2.15**  $\text{lfin.dim} \Lambda = \text{rfin.dim} \Lambda = \text{lFin.dim} \Lambda = \text{rFin.dim} \Lambda = \text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda$  for an Auslander artin ring  $\Lambda$ .

*Proof.* By Corollary 2.7 and Corollary 2.14 and its dual statement. ■

Recall that  $\Lambda$  has dominant dimension greater than or equal to  $k$  if each  $I_i$  is flat for any  $0 \leq i \leq k - 1$ .

The following are three important homological conjectures in representation theory of algebras.

*Nakayama Conjecture*<sup>[1]</sup>: An artin algebra  $\Lambda$  is self-injective if  $\Lambda$  has infinite dominant dimension.

It is easy to see that the Nakayama conjecture is a special case of the following conjecture.

*Auslander-Reiten Conjecture*<sup>[3]</sup>: An Auslander artin algebra has finite left and right self-injective dimensions.

*Finitistic Dimension Conjecture*<sup>[5,21]</sup>:  $\text{lfin.dim} \Lambda$  is finite for an artin algebra  $\Lambda$ .

From Corollary 2.14 we know that Auslander-Reiten Conjecture and Finitistic Dimension Con-

jecture are equivalent for Auslander artin algebras. So we get the following implications among these conjectures: Finitistic Dimension Conjecture  $\Rightarrow$  Auslander-Reiten Conjecture  $\Rightarrow$  Nakayama Conjecture. Furthermore, we get some equivalent conditions of Nakayama Conjecture as follows.

**Corollary 2.16** *Let  $\Lambda$  be an artin algebra with infinite dominant dimension. Then the following statements are equivalent.*

- (1)  $\Lambda$  is self-injective, that is, Nakayama Conjecture holds.
- (2)  $\text{lfin.dim}\Lambda = 0$ .
- (3) The right annihilator of any proper left ideal of  $\Lambda$  is always non-zero.
- (4) The right annihilator of any maximal left ideal of  $\Lambda$  is always non-zero.
- (5) For a module  $M$  in  $\text{mod } \Lambda$ ,  $M = 0$  if  $\text{grade}M = \infty$ .
- (6) For a simple module  $S$  in  $\text{mod } \Lambda$ ,  $S = 0$  if  $\text{grade}S = \infty$ .

*Proof.* The equivalence of (1) and (2) follows from Corollary 2.14. By Corollary 2.15 and the dual statements of [5] Corollary 5.6 and Theorem 5.4 we have that (2) and (3) are equivalent and that (2) implies (5). That (3) implies (4) and (5) implies (6) are trivial. Assume that  $J$  is a proper left ideal of  $\Lambda$ . Then  $J$  is contained in a maximal left ideal  $L$ . So the right annihilator of  $L$  is contained in the right annihilator of  $J$  and hence we have (4) implies (3). In the following we only need to prove (6) implies (4).

Suppose (6) holds and  $S$  is a simple module in  $\text{mod } \Lambda^{op}$  with  $S^* = 0$ . We claim that  $S = 0$ . Put  $K_i = \text{Ker}(I_i \rightarrow I_{i+1})$  for any  $i \geq 0$ . We then have an exact sequence:

$$0 \rightarrow \text{Hom}_\Lambda(S, K_i) \rightarrow \text{Hom}_\Lambda(S, I_i) \rightarrow \text{Hom}_\Lambda(S, K_{i+1}) \rightarrow \text{Ext}_\Lambda^{i+1}(S, \Lambda) \rightarrow 0$$

for any  $i \geq 0$ . Since  $\Lambda$  has infinite dominant dimension, each  $I_i$  is projective for all  $i \geq 0$ . So  $S^* = 0$  implies that  $\text{Hom}_\Lambda(S, I_i) = 0$  and  $\text{Hom}_\Lambda(S, K_i) = 0$  for any  $i \geq 0$ . Then by the exactness of the above sequence, we get that  $\text{Ext}_\Lambda^i(S, \Lambda) = 0$  for any  $i \geq 0$ , that is,  $\text{grade}S = \infty$ , and so we have  $S = 0$  by (6). Assume that  $L$  is a maximal left ideal of  $\Lambda$ . Then  $\Lambda/L$  is a simple module in  $\text{mod } \Lambda$ . By [5] (4.1) we have that the right annihilator of  $L$  is isomorphic to  $(\Lambda/L)^*$ . Now it is easy to verify (4) by the above claim. We are done. ■

The generalized Nakayama conjecture, posed by Auslander and Reiten in 1975, has an equivalent version as follows. A simple module  $S$  in  $\text{mod } \Lambda$  with  $\text{grade}S = \infty$  is zero for an artin algebra  $\Lambda$ . It is well known that Generalized Nakayama Conjecture  $\Rightarrow$  Nakayama Conjecture (see [1]). Corollary 2.16 not only gives another proof of this implication, but also claims that in order to verify Nakayama Conjecture it suffices to verify Generalized Nakayama Conjecture for artin algebras with infinite dominant dimension. Notice that the symmetric result of Corollary 2.16 is also true. So, if Nakayama

Conjecture holds, it turns out by Corollary 2.16 that an artin algebra with infinite dominant dimension has non-zero left and right zero divisors.

From Corollary 2.16 we in addition know that Nakayama Conjecture is true if and only if such an artin algebra  $\Lambda$  with the property that  $\Lambda$  has infinite dominant dimension as well as the right annihilator of some maximal left ideal of  $\Lambda$  is zero doesn't exist. The following result provides some partial support to Nakayama Conjecture, which says that such an artin algebra  $\Lambda$  with the property that  $\Lambda$  has infinite dominant dimension as well as the right annihilator of any maximal left ideal of  $\Lambda$  is zero doesn't exist.

**Corollary 2.17** *Let  $\Lambda$  be an artin algebra. If the right annihilator of any maximal left ideal of  $\Lambda$  is zero (especially,  $\Lambda$  has no non-zero right zero divisors), then the dominant dimension of  $\Lambda$  is finite.*

*Proof.* Let  $L$  be any left maximal ideal of  $\Lambda$ . Since the right annihilator of  $L$  is isomorphic to  $(\Lambda/L)^*$ , by assumption we then have  $(\Lambda/L)^* = 0$ .

If  $\Lambda$  has infinite dominant dimension (it certainly is an Auslander ring), then  $\text{s.gradeExt}_\Lambda^1(\Lambda/L, \Lambda) \geq 1$  and  $[\text{Ext}_\Lambda^1(\Lambda/L, \Lambda)]^* = 0$ . By using a similar argument to that in proving (6)  $\Rightarrow$  (4) of Corollary 2.16 we have  $\text{gradeExt}_\Lambda^1(\Lambda/L, \Lambda) = \infty$ . It follows from [12] Lemma 6.2 that  $\text{Ext}_\Lambda^1(\Lambda/L, \Lambda) = 0$  and so  $\Lambda$  is self-injective. Now by Corollary 2.16 we then have that the right annihilator of any proper (maximal) left ideal of  $\Lambda$  is always non-zero, which contradicts with the assumption. ■

### 3. Some useful properties on grade of modules

An Auslander ring is called Auslander-Gorenstein (resp. Auslander regular) if it has finite left and right self-injective dimensions (resp. finite global dimension). Let  $\Lambda$  be Auslander-Gorenstein. A module  $M$  in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) is called pure if  $\text{grade} N = \text{grade} M$  for any non-zero submodule of  $M$ . Björk raised in [7] p.144 an open question as follows. For an Auslander-Gorenstein ring  $\Lambda$ , is it true that  $\text{Ext}_\Lambda^{\text{grade} M}(M, \Lambda)$  is pure for each  $M \in \text{mod } \Lambda$ ? He showed in this reference that for an Auslander-Gorenstein ring  $\Lambda$  a module  $M$  in  $\text{mod } \Lambda$  (resp.  $\text{mod } \Lambda^{op}$ ) is pure if and only if  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(M, \Lambda), \Lambda) = 0$  for every  $i \neq \text{grade} M$ . By using this result he then answered the question affirmatively when  $\Lambda$  is Auslander regular or  $\Lambda$  is commutative. In this section, we will give a complete affirmative answer to this question.

**Lemma 3.1** *Let  $k$  be a positive integer. If  $\text{grade}_\Lambda^i(X, \Lambda) \geq i$  for any  $X \in \text{mod } \Lambda$  and  $1 \leq i \leq k$ , then  $\text{s.grade}_\Lambda^{i+1}(Y, \Lambda) \geq i$  for any  $Y \in \text{mod } \Lambda^{op}$  and  $1 \leq i \leq k$ .*

*Proof.* By the dual statements of [4] Theorem 1.7 and Proposition 4.2(b). ■

The following lemma plays a key role in proving the main result of this section.

**Lemma 3.2** *If  $\Lambda$  satisfies the condition that  $\text{grade}_\Lambda^i(X, \Lambda) \geq i$  for any  $X \in \text{mod } \Lambda$  and  $i \geq 1$ , then, for any  $M \in \text{mod } \Lambda$  with  $\text{grade} M$  finite, we have  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^{\text{grade} M}(M, \Lambda), \Lambda), \Lambda) = 0$  if and only if  $i \neq \text{grade} M$ .*

*Proof.* By assumption and Lemma 3.1 we have  $\text{grade}_\Lambda^{i+1}(Y, \Lambda) \geq i$  for any  $Y \in \text{mod } \Lambda^{op}$  and  $i \geq 1$ . Let  $M$  be in  $\text{mod } \Lambda$  and

$$\dots \rightarrow P_i \xrightarrow{f_i} \dots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \rightarrow M \rightarrow 0 \quad (1)$$

a projective resolution of  $M$  in  $\text{mod } \Lambda$ . Put  $t = \text{grade} M$  and  $X_i = \text{Coker } f_i^*$  for any  $i \geq 1$ .

If  $t = 0$ , that is,  $M^* \neq 0$ , then it is trivial that  $M^{***} \neq 0$  for  $M^*$  is isomorphic to a submodule of  $M^{***}$ . On the other hand, for any  $i \geq 1$ , we have  $\text{Ext}_\Lambda^i(M^*, \Lambda) \cong \text{Ext}_\Lambda^{i+2}(X_1, \Lambda)$ . So  $\text{grade} \text{Ext}_\Lambda^i(M^*, \Lambda) = \text{grade} \text{Ext}_\Lambda^{i+2}(X_1, \Lambda) \geq i + 1$ , and hence  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(M^*, \Lambda), \Lambda) = 0$ .

Now suppose  $t \geq 1$ . Then from the exact sequence (1) we get the following exact sequence:

$$0 \rightarrow M^* (= 0) \rightarrow P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} \dots \xrightarrow{f_t^*} P_t^* \rightarrow X_t \rightarrow 0,$$

which induces  $\text{r.pd}_\Lambda X_t \leq t$  and an exact sequence:

$$P_t^{**} \xrightarrow{f_t^{**}} \dots \xrightarrow{f_2^{**}} P_1^{**} \xrightarrow{f_1^{**}} P_0^{**} \rightarrow \text{Ext}_\Lambda^t(X_t, \Lambda) \rightarrow 0.$$

So we have  $M \cong \text{Ext}_\Lambda^t(X_t, \Lambda)$ .

By [14] Lemma 2 we have an exact sequence:

$$0 \rightarrow \text{Ext}_\Lambda^t(M, \Lambda) \rightarrow X_t \rightarrow P_{t+1}^* \rightarrow X_{t+1} \rightarrow 0 \quad (2)$$

Put  $Y_t = \text{Im}(X_t \rightarrow P_{t+1}^*)$ . Notice that  $\text{grade}_\Lambda^t(M, \Lambda) \geq t$ , we then get an exact sequence:

$$\begin{aligned} 0 = \text{Ext}_\Lambda^{t-1}(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) &\rightarrow \text{Ext}_\Lambda^t(Y_t, \Lambda) \rightarrow \text{Ext}_\Lambda^t(X_t, \Lambda) \rightarrow \text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) \\ &\rightarrow \text{Ext}_\Lambda^{t+1}(Y_t, \Lambda) \rightarrow \text{Ext}_\Lambda^{t+1}(X_t, \Lambda) = 0 \end{aligned} \quad (3)$$

and the isomorphisms:  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) \cong \text{Ext}_\Lambda^{i+1}(Y_t, \Lambda)$  for any  $i \geq t + 1$  and  $\text{Ext}_\Lambda^{i+1}(Y_t, \Lambda) \cong \text{Ext}_\Lambda^{i+2}(X_{t+1}, \Lambda)$  for any  $i \geq 0$ . So from the exact sequence (3) we get the following exact sequence:

$$0 \rightarrow \text{Ext}_\Lambda^{(t+1)}(X_{t+1}, \Lambda) \rightarrow M \rightarrow \text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) \rightarrow \text{Ext}_\Lambda^{t+2}(X_{t+1}, \Lambda) = 0 \quad (4)$$

and an isomorphism:

$$\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) \cong \text{Ext}_\Lambda^{i+2}(X_{t+1}, \Lambda) \quad (5)$$

for any  $i \geq t + 1$ .

From the isomorphism (5) we know that  $\text{grade} \text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) = \text{grade} \text{Ext}_\Lambda^{i+2}(X_{t+1}, \Lambda) \geq i + 1$  for any  $i \geq t + 1$ . It follows that  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda), \Lambda) = 0$  for any  $i \geq t + 1$ . On the

other hand,  $\text{grade}_\Lambda^t(M, \Lambda) \geq t$  also by assumption,  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) = 0$  for any  $0 \leq i \leq t-1$ . So we conclude that  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda), \Lambda) = 0$  for any  $i \neq t$ .

Since  $\text{grade}_\Lambda^{t+1}(X_{t+1}, \Lambda) \geq t$  and  $\text{grade}_\Lambda^{t+2}(X_{t+1}, \Lambda) \geq t+1$ , from the exact sequence (4) we get  $\text{Ext}_\Lambda^{t-1}(\text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda), \Lambda) \cong \text{Ext}_\Lambda^{t-1}(M, \Lambda) = 0$  for  $\text{grade} M = t$ . We now claim that  $\text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda), \Lambda) \neq 0$ . Otherwise, we have  $\text{grade} \text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) \geq t+1$  by the argument above. Since  $\text{grade} \text{Ext}_\Lambda^t(M, \Lambda) \geq t$  by assumption, it follows from [12] Lemma 6.2 that  $\text{Ext}_\Lambda^t(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) = 0$ . Notice that  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^t(M, \Lambda), \Lambda) = 0$  for any  $0 \leq i \leq t-1$  by assumption. Again by [12] Lemma 6.2 we then get  $\text{Ext}_\Lambda^t(M, \Lambda) = 0$ , which is a contradiction for  $\text{grade} M = t$ . The proof is finished. ■

We in addition need two lemmas. The following lemma was proved by Björk in Proposition 1.6 of [7] when  $\Lambda$  is Auslander-Gorenstein.

**Lemma 3.3** *Let  $\Lambda$  be an Auslander ring and  $M$  in  $\text{mod } \Lambda$  with  $\text{grade} M$  finite. Then  $\text{grade} \text{Ext}_\Lambda^{\text{grade} M}(M, \Lambda) = \text{grade} M$ .*

*Proof.* Suppose  $\text{grade} M = k (< \infty)$ . Since  $\Lambda$  is an Auslander ring,  $\text{grade} \text{Ext}_\Lambda^k(M, \Lambda) \geq k$ . We claim that  $\text{grade} \text{Ext}_\Lambda^k(M, \Lambda) = k$ . Otherwise, if  $\text{grade} \text{Ext}_\Lambda^k(M, \Lambda) > k$ , then  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^k(M, \Lambda), \Lambda) = 0$  for any  $0 \leq i \leq k$ . Thus  $\text{Ext}_\Lambda^k(M, \Lambda) = 0$  by [12] Lemma 6.2 and therefore  $\text{grade} M \geq k+1$ , which is a contradiction. ■

We use

$$0 \rightarrow \Lambda \rightarrow I'_0 \rightarrow I'_1 \rightarrow \cdots \rightarrow I'_i \rightarrow \cdots$$

to denote a minimal injective resolution of  $\Lambda$  as a left  $\Lambda$ -module. For a non-negative integer  $n$ , we use  $\mathcal{C}_\Lambda^n$  to denote the full subcategory of  $\text{mod } \Lambda$  consisting of the modules  $M$  with  $\text{Hom}_\Lambda(M, \bigoplus_{i=0}^n I'_i) = 0$  (see [8]).

**Lemma 3.4** *Let  $\Lambda$  be an Auslander-Gorenstein ring and  $k$  a positive integer. Then the following statements are equivalent for a module  $M$  in  $\text{mod } \Lambda$ .*

- (1)  $M$  is pure of grade  $k$ .
- (2)  $M$  is in  $\mathcal{C}_\Lambda^{k-1}$ , but not in  $\mathcal{C}_\Lambda^k$ .
- (3)  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(M, \Lambda), \Lambda) = 0$  for every  $i \neq k (= \text{grade} M)$ .

*Proof.* By [8] Lemma 1.1, for a non-negative integer  $n$  we have that a module  $M$  in  $\text{mod } \Lambda$  is in  $\mathcal{C}_\Lambda^n$  if and only if  $\text{s.grade} M \geq n+1$ . From this fact it is easy to get the equivalence of (1) and (2). The equivalence of (1) and (3) has been proved in [7] Proposition 1.9. ■

**Theorem 3.5** *Let  $\Lambda$  be a noetherian algebra and  $M$  a module in  $\text{mod } \Lambda$  with  $\text{grade} M$  finite. If the left flat dimension of  $I'_i$  is less than or equal to  $i+1$  for any  $i \geq 0$ , then we have*

$\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda), \Lambda), \Lambda) = 0$  if and only if  $i \neq \text{grade}M$ .

*Proof.* By the dual statement of [4] Theorem 4.7, for a noetherian algebra  $\Lambda$  we have that the left flat dimension of  $I_i'$  is less than or equal to  $i + 1$  for any  $i \geq 0$  if and only if  $\text{grade}_\Lambda^i(X, \Lambda) \geq i$  for any  $X \in \text{mod } \Lambda$  and  $i \geq 1$ . So our conclusion follows immediately from Lemma 3.2. ■

*Remark.* A  $k$ -Gorenstein algebra clearly satisfies the condition of Theorem 3.5. However, the interesting property of the left-right symmetry enjoyed by  $k$ -Gorenstein rings fails for this condition (see [13] for an example due to Hoshino).

By Auslander's Theorem and Lemma 3.2 we state the main result of this section as follows.

**Theorem 3.6** *Let  $\Lambda$  be an Auslander ring and  $M$  a module in  $\text{mod } \Lambda$  with  $\text{grade}M$  finite. Then  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda), \Lambda), \Lambda) = 0$  if and only if  $i \neq \text{grade}M$ .*

The following result is an immediate corollary of Theorem 3.6, which answers affirmatively the question mentioned in the beginning of this section.

**Corollary 3.7** *Let  $\Lambda$  be an Auslander-Gorenstein ring. Then  $\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda)$  is pure for any  $M \in \text{mod } \Lambda$ .*

*Proof.* Suppose  $M \in \text{mod } \Lambda$ . Since  $\Lambda$  has finite self-injective dimensions,  $\text{grade}M$  is finite. By Lemma 3.3, we have  $\text{grade}\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda) = \text{grade}M$ . On the other hand, by Theorem 3.6, we have  $\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^i(\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda), \Lambda), \Lambda) = 0$  if and only if  $i \neq \text{grade}M$ . It then follows from Lemma 3.4 that  $\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda)$  is pure. ■

**Corollary 3.8** ([7] Proposition 1.11) *Let  $\Lambda$  be an Auslander regular ring. Then  $\text{Ext}_\Lambda^{\text{grade}M}(M, \Lambda)$  is pure for any  $M \in \text{mod } \Lambda$ .*

In this following, we further give some useful properties of  $k$ -Gorenstein rings and Auslander rings, or concerning these rings.

Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequence in  $\text{mod } \Lambda$ . In general we have  $\text{grade}M_2 \geq \min\{\text{grade}M_1, \text{grade}M_3\}$ . Björk showed in [7] Proposition 1.8 that the equality holds if  $\Lambda$  is an Auslander-Gorenstein ring. We claim that the condition of  $\Lambda$  being Gorenstein is not necessary for this Björk's result, that is, we have

**Proposition 3.9** *Let  $\Lambda$  be an Auslander ring and  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  an exact sequence in  $\text{mod } \Lambda$ . Then  $\text{grade}M_2 = \min\{\text{grade}M_1, \text{grade}M_3\}$ .*

*Proof.* It suffices to prove  $\text{grade}M_2 \leq \min\{\text{grade}M_1, \text{grade}M_3\}$ . Put  $n = \min\{\text{grade}M_1, \text{grade}M_3\}$ . Without loss of generality, we assume  $n < \infty$ .

*Case I.*  $n = \text{grade}M_1 = \text{grade}M_3$ .

Consider the exact sequence:

$$\text{Ext}_\Lambda^{n-1}(M_1, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_3, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_2, \Lambda).$$

Since  $\text{grade} M_1 = n$ ,  $\text{Ext}_\Lambda^{n-1}(M_1, \Lambda) = 0$ . If  $\text{Ext}_\Lambda^n(M_2, \Lambda) = 0$ , then  $\text{Ext}_\Lambda^n(M_3, \Lambda) = 0$  and  $\text{grade} M_3 \geq n+1$ , which is a contradiction. So  $\text{Ext}_\Lambda^n(M_2, \Lambda) \neq 0$  and  $\text{grade} M_2 \leq n$ .

*Case II.*  $n = \text{grade} M_3 < \text{grade} M_1$ .

Consider the exact sequence:

$$\text{Ext}_\Lambda^{n-1}(M_1, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_3, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_2, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_1, \Lambda).$$

By assumption, we have  $\text{Ext}_\Lambda^{n-1}(M_1, \Lambda) = 0 = \text{Ext}_\Lambda^n(M_1, \Lambda)$  and  $\text{Ext}_\Lambda^n(M_3, \Lambda) \neq 0$ , so  $\text{Ext}_\Lambda^n(M_2, \Lambda) \neq 0$  and  $\text{grade} M_2 \leq n$ .

*Case III.*  $n = \text{grade} M_1 < \text{grade} M_3$ .

Consider the exact sequence:

$$\text{Ext}_\Lambda^n(M_2, \Lambda) \rightarrow \text{Ext}_\Lambda^n(M_1, \Lambda) \rightarrow \text{Ext}_\Lambda^{n+1}(M_3, \Lambda).$$

If  $\text{Ext}_\Lambda^n(M_2, \Lambda) = 0$ , then  $\text{Ext}_\Lambda^n(M_1, \Lambda)$  is isomorphic to a submodule of  $\text{Ext}_\Lambda^{n+1}(M_3, \Lambda)$ . Since  $\Lambda$  is an Auslander ring,  $\text{grade} \text{Ext}_\Lambda^n(M_1, \Lambda) \geq n+1$ . By assumption and [12] Lemma 6.2 we then have  $\text{Ext}_\Lambda^n(M_1, \Lambda) = 0$  and  $\text{grade} M_1 \geq n+1$ , which is a contradiction. So  $\text{Ext}_\Lambda^n(M_2, \Lambda) \neq 0$  and  $\text{grade} M_2 \leq n$ . ■

We use  $\mathcal{T}^k(\text{mod } \Lambda)$  to denote the full subcategory of  $\text{mod } \Lambda$  consisting of  $k$ -torsionfree modules. The extension closure of  $\mathcal{T}^i(\text{mod } \Lambda)$  for each  $1 \leq i \leq k$  gives an equivalent characterization of a noetherian algebra with the property of  $\text{r.f.d.}_\Lambda I_i \leq i+1$  for each  $0 \leq i \leq k-1$  (see [4] and [13]).

**Proposition 3.10** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequence in  $\text{mod } \Lambda$ . For a positive integer  $k$ , if  $\text{grade} C \geq k$  where  $C = \text{Coker}(M_2^* \rightarrow M_1^*)$ , then we have*

- (1) *If  $M_2 \in \mathcal{T}^{k+1}(\text{mod } \Lambda)$  and  $M_3 \in \mathcal{T}^k(\text{mod } \Lambda)$ , then  $M_1 \in \mathcal{T}^{k+1}(\text{mod } \Lambda)$ .*
- (2) *If  $M_1, M_3 \in \mathcal{T}^k(\text{mod } \Lambda)$ , then  $M_2 \in \mathcal{T}^k(\text{mod } \Lambda)$ .*
- (3) *If  $M_1 \in \mathcal{T}^k(\text{mod } \Lambda)$  and  $M_2 \in \mathcal{T}^{k-1}(\text{mod } \Lambda)$ , then  $M_3 \in \mathcal{T}^{k-1}(\text{mod } \Lambda)$ .*

*Proof.* Consider the following commutative diagram with exact rows and columns and last two rows splitting:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 \oplus G_0 & \longrightarrow & G_0 \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & F_1 & \longrightarrow & F_1 \oplus G_1 & \longrightarrow & G_1 \longrightarrow 0
\end{array}$$

where all  $F_i$  and  $G_i$  are projective. Then we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_3^* & \longrightarrow & M_2^* & \longrightarrow & M_1^* \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_0^* & \longrightarrow & G_0^* \oplus F_0^* & \longrightarrow & F_0^* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_1^* & \longrightarrow & G_1^* \oplus F_1^* & \longrightarrow & F_1^* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & X & & Y & & Z \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

It follows from the snake lemma that  $0 \rightarrow M_3^* \rightarrow M_2^* \rightarrow M_1^* \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is exact, which can be splitted into two short exact sequences:  $0 \rightarrow C \rightarrow X \rightarrow K \rightarrow 0$  and  $0 \rightarrow K \rightarrow Y \rightarrow Z \rightarrow 0$ , where  $C = \text{Coker}(M_2^* \rightarrow M_1^*)$  and  $K = \text{Im}(X \rightarrow Y)$ . Then we get two long exact sequences:

$$\text{Ext}_\Lambda^i(K, \Lambda) \rightarrow \text{Ext}_\Lambda^i(X, \Lambda) \rightarrow \text{Ext}_\Lambda^i(C, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(K, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(X, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(C, \Lambda)$$

and

$$\text{Ext}_\Lambda^i(Z, \Lambda) \rightarrow \text{Ext}_\Lambda^i(Y, \Lambda) \rightarrow \text{Ext}_\Lambda^i(K, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(Z, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(Y, \Lambda) \rightarrow \text{Ext}_\Lambda^{i+1}(K, \Lambda)$$

for any  $i \geq 0$ .

If  $M_2 \in \mathcal{T}^{k+1}(\text{mod } \Lambda)$  and  $M_3 \in \mathcal{T}^k(\text{mod } \Lambda)$  and  $\text{grade } C \geq k$ , then  $Y$  is a  $\mathbf{W}^{k+1}$ -module,  $X$  is a  $\mathbf{W}^k$ -module and  $C^* = 0$  and  $\text{Ext}_\Lambda^i(C, \Lambda) = 0$  for any  $1 \leq i \leq k-1$ . From the first long exact sequence we have that  $\text{Ext}_\Lambda^i(K, \Lambda) = 0$  for any  $1 \leq i \leq k$ . The exactness of the second long sequence then

yields that  $\text{Ext}_\Lambda^i(Z, \Lambda) = 0$  for any  $2 \leq i \leq k+1$ . But  $M_1$  is clearly torsionless, so  $\text{Ext}_\Lambda^1(Z, \Lambda) = 0$  and  $Z$  is a  $W^{k+1}$ -module, which implies that  $M_1$  is  $(k+1)$ -torsionfree. This finishes the proof of (1). Similarly we get (2) and (3). ■

In the following we give some examples of rings satisfying the grade condition in Proposition 3.10.

**Example 3.11** (1) From the proof of [13] Theorem 2.2 we know that if the right flat dimension of  $\oplus_{i=0}^{k-1} I_i$  is less than or equal to  $k$  then the grade condition in Proposition 3.10 is satisfied. So, if  $\text{r.fd}_\Lambda I_i \leq i+1$  for any  $0 \leq i \leq k-1$ , especially, if  $\Lambda$  is a  $k$ -Gorenstein ring or  $\Lambda$  has dominant dimension greater than or equal to  $k$ , the grade condition is also satisfied.

(2) By [16] Proposition 1 we have  $\text{l.id}_\Lambda \Lambda = \sup\{\text{the flat dimension of any right injective } \Lambda\text{-module}\}$ . Thus this grade condition is satisfied if  $\text{l.id}_\Lambda \Lambda \leq k$ , and therefore, in case  $\text{l.id}_\Lambda \Lambda = \text{r.id}_\Lambda \Lambda \leq k$ ,  $\mathcal{T}^k(\text{mod } \Lambda)$  is a resolving subcategory of  $\text{mod } \Lambda$  (that is,  $\mathcal{T}^k(\text{mod } \Lambda)$  is closed under extensions, kernels of epimorphisms and contains projective modules).

#### 4. When is a torsionless module reflexive?

Let  $A$  be a torsionless module in  $\text{mod } \Lambda$ . Then  $A$  can be embedded into a finitely generated free module  $G$ . We use  $\mathcal{C}$  to denote the subcategory of  $\text{mod } \Lambda$  consisting of the non-zero modules  $C$  such that there is an exact sequence  $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$  in  $\text{mod } \Lambda$  with  $G$  free. From now on we will keep this notation.

**Proposition 4.1** *If  $\Lambda$  has the property that the right flat dimension of  $I_i$  is less than or equal to  $i+1$  for any  $0 \leq i \leq k-1$ , then, for any  $t$ -torsionfree module  $A$  (where  $1 \leq t \leq k$ ) in  $\text{mod } \Lambda$  and  $C$  in  $\mathcal{C}$ , there is an exact sequence  $0 \rightarrow F \rightarrow T \rightarrow C \rightarrow 0$  in  $\text{mod } \Lambda$  with  $F$  free and  $T$   $(t-1)$ -torsionfree.*

*Proof.* Under the assumption of this proposition, we have that for any  $1 \leq t \leq k$  a module in  $\text{mod } \Lambda$  is  $t$ -torsionfree if and only if it is  $t$ -syzygy by [4] Proposition 1.6 and Theorem 1.7. So for a  $t$ -torsionfree module  $A$  in  $\text{mod } \Lambda$ , there is an exact sequence  $0 \rightarrow A \rightarrow F \rightarrow K \rightarrow 0$  in  $\text{mod } \Lambda$  with  $F$  free and  $K$   $(t-1)$ -torsionfree. Let  $C \in \mathcal{C}$ . Then there is an exact sequence  $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$  in  $\text{mod } \Lambda$  with  $G$  free. Consider the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & F & \longrightarrow & T & \longrightarrow & C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & K & \xlongequal{\quad} & K & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

Under the assumption, we in addition have that for any  $1 \leq t \leq k$  the subcategory of  $\text{mod } \Lambda$  consisting of  $t$ -torsionfree modules is extension closed by [4] Theorem 1.7. So from the exactness of the middle column in above diagram,  $T$  is  $(t-1)$ -torsionfree. Thus the middle row in above diagram is the desired exact sequence. ■

Recall from [8] that a module  $M$  in  $\text{mod } \Lambda$  is said to be pseudo-null if  $M \in \mathcal{C}_\Lambda^1$  (that is,  $\text{Hom}_\Lambda(M, I'_0 \oplus I'_1) = 0$ ).

**Proposition 4.2** *Let  $\Lambda$  be a 2-Gorenstein ring,  $A$  a torsionless module and  $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$  an exact sequence in  $\text{mod } \Lambda$  with  $G$  free and  $C \in \mathcal{C}$ . Then we have*

- (1)  $\text{Coker} \sigma_A$  is pseudo-null.
- (2)  $A^{**}$  is isomorphic to a submodule of  $G$ .
- (3) If  $C$  has no non-zero pseudo-null submodule, then  $A$  is reflexive.

*Proof.* (1) By [15] Lemma 2.1 there is a module  $X$  in  $\text{mod } \Lambda^{op}$  such that  $\text{Coker} \sigma_A \cong \text{Ext}_\Lambda^2(X, \Lambda)$ . Since  $\Lambda$  is 2-Gorenstein,  $\text{Hom}_\Lambda(\text{Ext}_\Lambda^2(X, \Lambda), I_0 \oplus I_1) = 0$  by [17] Proposition 3.

- (2) From the exact sequence  $0 \rightarrow A \rightarrow G \rightarrow C \rightarrow 0$  we get a long exact sequence:

$$0 \rightarrow C^* \rightarrow G^* \rightarrow A^* \rightarrow \text{Ext}_\Lambda^1(C, \Lambda) \rightarrow 0.$$

Put  $K = \text{Im}(G^* \rightarrow A^*)$ . Then we get two exact sequences  $0 \rightarrow K^* \rightarrow G^{**} (\cong G)$  and  $[\text{Ext}_\Lambda^1(C, \Lambda)]^* \rightarrow A^{**} \rightarrow K^*$ . Because  $\Lambda$  is 2-Gorenstein,  $\text{s.grade} \text{Ext}_\Lambda^1(C, \Lambda) \geq 1$  and  $[\text{Ext}_\Lambda^1(C, \Lambda)]^* = 0$ . Then the assertion follows.

- (3) Notice that  $A^{**}/A$  is pseudo-null by (1) and  $A^{**}/A$  is isomorphic to a submodule of  $C$  by (2), so  $A^{**}/A = 0$  by assumption and hence  $A \cong A^{**}$ . Since  $\Lambda$  is 2-Gorenstein,  $[\text{Ext}_\Lambda^2(-, \Lambda)]^*$  vanishes on  $\text{mod } \Lambda$ . It then follows from [12] Lemma 1.6 that  $A$  is reflexive. ■

We now give a criterion for judging when a torsionless module is reflexive.

**Theorem 4.3** *Let  $\Lambda$  be a 2-Gorenstein ring and  $A$  a torsionless module in  $\text{mod } \Lambda$ . Then the following statements are equivalent.*

- (1)  *$A$  is reflexive.*
- (2)  *$C$  has no non-zero pseudo-null submodule for any  $C \in \mathcal{C}$ .*
- (3)  *$C$  has no non-zero pseudo-null submodule for some  $C \in \mathcal{C}$ .*

*Proof.* That (2) implies (3) is trivial and (3) implies (1) follows from Proposition 4.2(3). So we only need to prove (1) implies (2). Assume that  $A$  is reflexive. Then, by Proposition 4.1, for any  $C \in \mathcal{C}$ , there is an exact sequence  $0 \rightarrow F \rightarrow T \rightarrow C \rightarrow 0$  in  $\text{mod } \Lambda$  with  $F$  free and  $T$  torsionless. By [19] Corollary 1.3 it is easy to see that  $C$  can be embedded into a finite direct summand of  $I'_0 \oplus I'_1$  and so  $C$  has no non-zero pseudo-null submodule. ■

Let  $\Lambda$  be an Auslander-Gorenstein ring and  $I$  a non-zero proper left ideal of  $\Lambda$ . Then, by Lemma 3.4, we have that  $\Lambda/I$  has no non-zero pseudo-null submodule if and only if  $\Lambda/I$  is pure of grade 1. It is showed in [8] Lemma 3.12 that if  $\Lambda$  is an Auslander regular ring without no non-zero zero divisors then  $I$  is reflexive if and only if  $\Lambda/I$  is pure of grade 1. The following immediate consequence of Theorem 4.3 generalizes this result.

**Corollary 4.4** *Let  $\Lambda$  be a 2-Gorenstein ring. Then a non-zero proper left ideal  $I$  of  $\Lambda$  is reflexive if and only if  $\Lambda/I$  has no non-zero pseudo-null submodule.*

Recall that a right  $\Lambda$ -module  $M$  is said to have an injective resolution with a redundant image from a positive integer  $n$  if the  $n$ th cosyzygy  $\Omega_n$  has a decomposition  $\Omega_n = \oplus_{i \in I} A_i$  such that each  $A_i$  is a direct summand of a cosyzygy  $\Omega_{\alpha_i}$  for some  $\alpha_i \neq n$ . It is showed in [10] Theorem 1 that if  $\Lambda_\Lambda$  has an injective resolution with a redundant image from  $n$  then  $\oplus_{i=0}^n I'_i$  is an injective cogenerator in the category of left  $\Lambda$ -module.

Assume that  $\Lambda$  is a 2-Gorenstein ring with the property of  $\Lambda_\Lambda$  having an injective resolution with a redundant image from 1. Then  $I'_0 \oplus I'_1$  is an injective cogenerator in the category of right  $\Lambda$ -module. So every module in  $\text{mod } \Lambda$  has no non-zero pseudo-null submodule and hence each torsionless module in  $\text{mod } \Lambda$  is reflexive by Theorem 4.3. It then follows from [18] Theorem 5.1 that  $\text{r.id}_\Lambda \Lambda \leq 1$ . Hence we have established the following result.

**Corollary 4.5** *Let  $\Lambda$  be a 2-Gorenstein ring. If  $\Lambda_\Lambda$  has an injective resolution with a redundant image from 1, then  $\text{r.id}_\Lambda \Lambda \leq 1$ .*

M. Ramras in [11] p.380 raised an open question: for a left and right noetherian ring  $\Lambda$ , when is each reflexive module in  $\text{mod } \Lambda$  projective? A further related question is: for a positive integer  $k$ , when is each  $k$ -torsionfree module in  $\text{mod } \Lambda$  projective? In the following we will deal with these questions and give some affirmative answers to them. We remark that it is trivial that each torsionless

module in  $\text{mod } \Lambda$  (or  $\text{mod } \Lambda^{op}$ ) is projective if and only if  $\Lambda$  is a hereditary ring (that is, the global dimension of  $\Lambda$  is less than or equal to one).

We first give the following

**Proposition 4.6** *The following statements are equivalent.*

- (1) *Each torsionless in  $\text{mod } \Lambda$  is projective.*
- (2) *Each  $W^1$ -module in  $\text{mod } \Lambda^{op}$  is projective.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $N$  be a  $W^1$ -module in  $\text{mod } \Lambda^{op}$  and  $Q_1 \xrightarrow{g} Q_0 \rightarrow N \rightarrow 0$  a projective resolution of  $N$  in  $\text{mod } \Lambda^{op}$ . Put  $M = \text{Coker } g^*$ . Then  $M$  is torsionless in  $\text{mod } \Lambda$  by [15] Lemma 2.1 and so  $M$  is projective by (1), which implies that  $M^*$  is projective and hence  $N^*$  and  $N^{**}$  are also projective. On the other hand, also by [15] Lemma 2.1 we have an exact sequence:

$$0 \rightarrow \text{Ext}_{\Lambda}^1(M, \Lambda) \rightarrow N \xrightarrow{\sigma_N} N^{**} \rightarrow \text{Ext}_{\Lambda}^2(M, \Lambda) \rightarrow 0.$$

Thus  $N$  is reflexive and therefore it is projective.

(2)  $\Rightarrow$  (1) Let  $M$  be a torsionless module in  $\text{mod } \Lambda$  and  $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$  a projective resolution of  $M$  in  $\text{mod } \Lambda$ . Put  $N = \text{Coker } f^*$ . Then  $N$  is a  $W^1$ -module in  $\text{mod } \Lambda^{op}$  by [15] Lemma 2.1 and so  $N$  is projective by (2). Thus we have that  $M^*$  and  $M^{**}$  are projective. On the other hand,  $M$  is reflexive again by [15] Lemma 2.1, we conclude that  $M$  is also projective. ■

We now give a new characterization of hereditary rings.

**Corollary 4.7**  *$\Lambda$  is hereditary if and only if each  $W^1$ -module in  $\text{mod } \Lambda$  (or  $\text{mod } \Lambda^{op}$ ) is projective.*

**Proposition 4.8** *For any  $k \geq 2$ , the following statements are equivalent.*

- (1) *Each  $k$ -torsionfree module in  $\text{mod } \Lambda$  is projective.*
- (2) *Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is projective.*

*Proof.* (1)  $\Rightarrow$  (2) Let  $N$  be a  $W^k$ -module in  $\text{mod } \Lambda^{op}$  and  $Q_1 \xrightarrow{g} Q_0 \rightarrow N \rightarrow 0$  a projective resolution of  $N$  in  $\text{mod } \Lambda^{op}$ . Put  $M = \text{Coker } g^*$ . It is easy to see that  $M^* \cong \text{Ker } g$ . It follows that  $M$  is  $k$ -torsionfree (and certainly, reflexive) in  $\text{mod } \Lambda$  so  $M$  is projective by (1). By using a similar argument to that of (1) implying (2) in Proposition 4.6, we have that  $N$  is projective.

(2)  $\Rightarrow$  (1) Let  $M$  be a  $k$ -torsionfree module in  $\text{mod } \Lambda$  and  $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$  a projective resolution of  $M$  in  $\text{mod } \Lambda$ . Put  $N = \text{Coker } f^*$ . Then  $M$  is reflexive and  $N$  is a  $W^k$ -module in  $\text{mod } \Lambda^{op}$  and so  $N$  is projective by (2). It follows that  $M^*$  and  $M^{**}$  are projective and consequently  $M$  is also projective. ■

The following is the main result of this section.

**Theorem 4.9** *Let  $k$  be a positive integer or infinite. Then the following statements are equivalent.*

- (1) Each  $k$ -torsionfree module in  $\text{mod } \Lambda$  is projective.
- (2) Each  $W^k$ -module in  $\text{mod } \Lambda^{op}$  is projective.

*Proof.* We have proved in Propositions 4.6 and 4.8 the case  $k$  is a positive integer. When  $k$  is infinite, the proof is similar to that of Proposition 4.8, we omit it. ■

**Corollary 4.10** *If the global dimension of  $\Lambda$  is less than or equal to  $k$ , then each  $k$ -torsionfree module in  $\text{mod } \Lambda$  is projective.*

*Proof.* Let  $N$  be a  $W^k$ -module in  $\text{mod } \Lambda^{op}$ . Since the global dimension of  $\Lambda$  is less than or equal to  $k$ , it follows that  $N$  is a  $W^\infty$ -module and  $\text{l.pd}_\Lambda N \leq k$ . Now by applying Lemma 2.9 we have that  $N$  is projective. We then get our conclusion by Theorem 4.9. ■

Especially we have the following

**Corollary 4.11** *If the global dimension of  $\Lambda$  is less than or equal to 2, then each reflexive module in  $\text{mod } \Lambda$  is projective.*

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